

Tavis-Cummings model beyond the rotating wave approximation: Inhomogeneous coupling

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We present the analytical solution of the Tavis-Cummings (TC) model for more than one qubit inhomogeneously coupled to a single mode radiation field beyond the rotating-wave approximation (RWA). The significant advantage of the displaced oscillator basis enables us to apply the same truncation techniques adopted in the single qubit Jaynes-Cummings (JC) model to the multiple qubits system. The derived analytical spectrum match perfectly the exact diagonalization numerical solutions of the inhomogeneous TC model in the parameter regime where the qubits transition frequencies are far off-resonance with the field frequency and the interaction strengths reach the ultra-strong coupling regime. The two-qubit TC model is quasi-exactly solvable because part of the spectra can be determined exactly in the homogeneous coupling case with two identical qubits or with symmetric(asymmetric) detuning. By means of the fidelity of quantum states we identify several nontrivial level crossing points in the same parity subspace, which implies that homogeneous coupled two-qubit TC model with $\omega_1 = \omega_2$ or $\omega_1 \pm \omega_2 = 2\omega_c$ is integrable. We further explore the time evolution of the qubit's population inversion and the entanglement behavior taking two qubits as an example. The analytical methods provide unexpectedly accurate results in describing the dynamics of the qubit in the present experimentally accessible coupling regime, showing that the collapse-revival phenomena emerge, survive, and are finally destroyed when the coupling strength increases beyond the ultra-strong coupling regime. The inhomogeneous coupling system exhibits new dynamics, which are different from homogeneous coupling case. The suggested procedure applies readily to the multiple qubits system such as the GHZ state entanglement evolution and quantum entanglement between a single photon and superconducting qubits of particular experiment interest.

PACS numbers: 42.50.Pq, 42.50.Md, 03.65.Ud

I. INTRODUCTION

The Jaynes-Cummings (JC) model with the rotating wave approximation (RWA), first introduced in 1963 [1], is the simplest model that describes the interaction between a two-level atom and a single mode quantized radiation field. The RWA is applicable when the applied electromagnetic field frequency ω_c is near resonance with the atom transition frequency ω_j and the interaction between the atom and the radiation field is weak. The reason is that the contribution of the counter-rotating terms of the system is very small. Typical new era of experiments witness the breakdown of the JC model in terms of both coupling and detuning, including circuit QED experiments with superconducting qubits coupled to LC and waveguide resonators [2–5] and Cooper-pair boxes or Josephson phase qubits coupled to nanomechanical resonators [6–10]. Each of these artificial atoms has an internal degree of freedom (d.o.f.) that can be either up or down, creating a spin-1/2 system. These systems generally allow coupling strengths up to $g_j/\omega_c \simeq 0.1$ in the so-called ultrastrong coupling regime, or the qubit transition frequency far-detuned from the field frequency [11–24]. An adiabatic approximation approach [15] was proposed to treat the parameter regime outside the near-resonance and weak-coupling assumption of RWA based

on the displaced oscillator basis. In this basis the Hamiltonian are truncated to a block-diagonal form and the blocks are solved individually. The time evolution of the two-level-system occupation probability with thermal and coherent state initial conditions for the oscillator exhibits clear signals of collapse and revival.

The quantum Rabi model [25], or JC model without the RWA, was recently declared solved exactly in [26, 27]. By means of the representation of bosonic operators in the Bargmann space Braak argued that the regular spectrum of the Rabi model was given by the zeros of a transcendental function, which is given as an infinite power series. Chen et. al. mapped the model to a polynomial equation with a single variable in terms of tunable extended bosonic coherent states [28]. They recover Braak's exact solution in an alternative more physical way and point out that both methods have one thing in common: the spectrum can not be obtained without truncation in the power series [29]. Thus the Rabi model is quasi-exactly solvable in the sense that only a finite part of the spectrum can be obtained in closed form and the remaining part of the spectrum can only be determined by numerical means [30–33]. The number of ca. 1350 calculable energy levels in each parity subspace are obtained in double precision by an elementary stepping algorithm up to two orders of magnitude higher than Braak's solution [31]. Quantum integrability is, according to Braak's criterion, equivalent to the existence of quantum numbers that classify eigenstates uniquely. The Rabi model is integrable because it has

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two d.o.f. and the eigenstates can be uniquely labelled by two quantum numbers associated with the energy and the parity, respectively. Moreover, examples of nonintegrable but (quasi)exactly solvable system are given with broken parity symmetry or vanished level splitting of an additional qubit [26, 33]. These theoretical progress has renewed the interest in the Rabi and related models. Analytical solutions of this model have brought clarity and intuition to several important problems and experimental results in contemporary quantum information. Furthermore, it is expected that the experiments could reach the deep strong coupling regime [34, 35] where the ratio of the coupling strength to the relevant frequencies exceeds unity. Perturbative methods and the concept of Rabi oscillations should be superseded by novel physics such as parity chains and photon number wave packets.

To describe the collective behavior of multiple atomic dipoles interacting with an electromagnetic field mode, the Dicke model [36] was introduced where the Pauli operators are summed and transformed into a bosonic operator. Though very successful in treating the system of an alkaline atomic ensemble [37] in an optical microcavity with the number of atoms over 10^5 , it does not apply well to the case of multiqubit superconducting circuits with $N \leq 10$. Theoretical studies for a finite number N of qubits in the system often employ the Tavis-Cummings (TC) model [38] under RWA, where all the spins are grouped into a total large spin. The TC model has received much attention and has been involved in both experiments and theories [39–49]. Being exactly solvable only when the coupling is homogeneous and when the eigenfrequencies between the qubits and the photon mode are equal, it was recently [41] extended beyond the RWA for quasi-degenerate qubits in the parameter regime in which the frequencies of the qubits are much smaller than the oscillator frequency and the coupling strength is allowed to be ultrastrong. The case of inhomogeneous coupling is drastically different - there is no such straightforward way to access the Hilbert space. The extension to the inhomogeneous coupling system is limited to RWA [43, 48] or numerical exact approach on the entanglement evolution of two independent JC atoms [40]. It is worthwhile to notice that more efforts are paid to the system composed of two nonidentical qubits [39], or the $N = 3$ Dicke model which couples three qubits to a single radiation mode and constitutes the simplest quantum-optical system allowing for Greenberger-Horne-Zeilinger (GHZ) states [49].

We in this paper solve the TC model beyond the RWA with N qubits coupled to a single oscillator mode by comprehensively considering the recently developed approaches in solving the JC model with extension to the case of inhomogeneous coupling. A systematic truncated method is developed in finding the exact wave functions of the model with N discrete and one continuous d.o.f. In the basis of the displaced operators we construct the Hamiltonian by primitive building blocks and allow transitions between adjacent blocks in addition. We take

mainly arbitrary two qubits interacting with a bose field as an example. The analytical eigensolutions are derived in the zeroth-order and first-order approximation to the exact wave function in deep strong coupling parameter regimes. The procedure is easily extended to systems with more than two qubits. We further examine the solvability and integrability of the system and level crossing points in the energy spectrum of the same parity space are related to a hidden symmetry in the system. Then, starting from any initial state of the system we are able to derive the time evolution properties of the qubits by using a linear combination of the analytical eigensolutions and tracing over the oscillator field. In other words, some general techniques are applied to investigate the time evolution of the rather complicated multiqubit-field system. Subsequently, we can apply the approximated eigensolutions to study the dynamical evolution of the entanglement between the two qubits as a fundamental consequence of quantum mechanics and as a resource for communication and information processing [50–54].

The paper is organized as follows. In Sec. II the analytical eigen solutions of the TC model beyond the RWA is given after introducing the general procedure of constructing the determinant of the secular equation. Sec. III is devoted to the dynamical behaviors of the qubits, in which the analytical eigensolutions are employed to approximately describe the time evolution of the probability of finding both qubits in their initial state and the population inversion when the quantum field is prepared initially in the displaced coherent state. In Sec. IV we further study the entanglement dynamics for two qubits starting from the Bell state and the field in a coherent state. Finally, we make some discussions on the characteristics of case of unequal coupling strengthes for the two qubits from the present study in Sec. V.

II. EIGEN SOLUTIONS OF THE TC MODEL BEYOND THE RWA

The system we consider here consists of N qubits inhomogeneously interacting with a single-mode bose field. It is described by the TC Hamiltonian beyond the RWA (we set $\hbar = 1$) [38]

$$H = \omega_c a^\dagger a + \sum_{j=1}^N \left(-\frac{\omega_j}{2} \hat{\sigma}_j^x + g_j (a^\dagger + a) \hat{\sigma}_j^z \right), \quad (1)$$

where a^\dagger (a) is the bosonic creation (annihilation) operator of the single bosonic mode with frequency ω_c , ω_j denotes the energy splitting of j -th qubit described by Pauli matrices $\hat{\sigma}_j^k$ ($k = x, y, z$), and g_j is the dipole-field coupling strength between the qubit j and the field. Here we have rotated the system around the y -axis by an angle $\pi/4$ realized through a unitary transformation [40] $V = \exp\left(\frac{i\pi}{4} \sum_j \hat{\sigma}_j^y\right)$. Basically the calculation here applies to arbitrary non-identical two-level atoms and non-uniform coupling strengths g_j in any form. We also note

that the Hamiltonian (1) conserves the global parity operator defined as $\Pi = \prod_{j=1}^N \hat{\sigma}_j^x \exp(i\pi a^\dagger a)$, i.e. $[H, \Pi] = 0$.

In the following we use the parity operator to decompose the Hilbert space into even and odd subspaces.

For the convenience of description, we take $N = 2$ as an illustrative example. Denote the upper and lower

eigenstates of $\hat{\sigma}_j^z$ as $|1\rangle_j$ and $|0\rangle_j$ respectively. Formally we treat qubit 2 as a new member to the Rabi model [25, 26] and unfold the dimension of the space from 2 to 4. In the basis of product space of the two qubits, i.e. $|11\rangle = |1\rangle_2 \otimes |1\rangle_1$, $|10\rangle = |1\rangle_2 \otimes |0\rangle_1$, $|01\rangle = |0\rangle_2 \otimes |1\rangle_1$, and $|00\rangle = |0\rangle_2 \otimes |0\rangle_1$, we may write the Hamiltonian into a matrix form

$$H = \begin{pmatrix} \omega_c (a^\dagger a + \beta_1 (a^\dagger + a)) & -\frac{\omega_1}{2} & 0 & 0 \\ -\frac{\omega_1}{2} & \omega_c (a^\dagger a + \beta_2 (a^\dagger + a)) & -\frac{\omega_2}{2} & 0 \\ -\frac{\omega_2}{2} & 0 & \omega_c (a^\dagger a + \beta_3 (a^\dagger + a)) & -\frac{\omega_1}{2} \\ 0 & -\frac{\omega_2}{2} & -\frac{\omega_1}{2} & \omega_c (a^\dagger a + \beta_4 (a^\dagger + a)) \end{pmatrix} \quad (2)$$

We notice that the original 2×2 Hamiltonian [22] as a primitive block is shifted along the diagonal line with g_1 and g_2 being recombined into 4 coupling parameters β_i with relations $\beta_1 = -\beta_4 = (g_2 + g_1)/\omega_c$ and $\beta_2 = -\beta_3 = (g_2 - g_1)/\omega_c$, while the transition frequency ω_2 always appears in the off-diagonal blocks of the matrix. Similar procedure can be applied when we add one more qubit to the system. The basis of N qubits is 2^N dimension and we have $2^{(N-1)}$ independent dimensionless coupling parameters β_i . A special case is the system of two identical qubits ($\omega_1 = \omega_2$) coupling with a common oscillator mode, which has been extended beyond the RWA in Ref. [41]. By assuming that the coupling parameters are larger than the transition frequencies, i.e. in the deep-strong-coupling regime, $g_1, g_2 \gg \omega_1, \omega_2$, the eigenvalues have been calculated up to the second-order perturbation correction [39]. The main results for these limiting cases may be readily reproduced from the method in this paper.

Let us first introduce the displacement operators [15, 56] $\hat{D}(\beta_i) = \exp[\beta_i (a^\dagger - a)]$, which will translate the field operators a^\dagger and a by a distance β_i and give rise to $A_i^\dagger = \hat{D}^\dagger(\beta_i) a^\dagger \hat{D}(\beta_i) = a^\dagger + \beta_i$ and $A_i = \hat{D}^\dagger(\beta_i) a \hat{D}(\beta_i) = a + \beta_i$, and will transform the number state $|n\rangle$ defined as $a^\dagger a |n\rangle = n |n\rangle$ into the so-called displaced Fock number state defined as $A_i^\dagger A_i |n\rangle_{A_i} = n |n\rangle_{A_i}$, i.e. $\hat{D}^\dagger(\beta_i) |n\rangle = |n\rangle_{A_i}$. The states $|n\rangle_{A_i}$ are orthogonal for the same index i and non-orthogonal for different subspaces i and j . The lack of orthogonality between states with different displacements leads to the unusual results in the dynamics of population inversion and entanglement which will be shown later in this work.

The diagonal elements of the Hamiltonian are in this way reconstructed as $\omega_c (A_i^\dagger A_i - \beta_i^2)$ with the off-diagonal ω_j unchanged. The Hilbert space of the diagonal Hamiltonian is now of the form of a combination of qubit basis and displaced oscillator basis, e.g. $|111\dots\rangle |n\rangle_{A_1}$, which can be taken as a starting point to

expand the eigenfunction of the total Hamiltonian H . For two-qubit system we suppose that

$$|\psi\rangle = \sum_{n=0}^{\infty} (d_{1n} |11\rangle |n\rangle_{A_1} + d_{2n} |10\rangle |n\rangle_{A_2} + d_{3n} |01\rangle |n\rangle_{A_3} + d_{4n} |00\rangle |n\rangle_{A_4}) \quad (3)$$

which in addition should be the eigenfunction of the parity operator Π , i.e. $\Pi|\psi\rangle = \kappa|\psi\rangle$ with $\kappa = +, -$ for even and odd parity respectively. Actually we find that the σ 's operators in Π transform the qubit basis from $|11\rangle$ ($|10\rangle$) to $|00\rangle$ ($|01\rangle$) and vice versa, while the field operators set up links between the displaced Fock state $|n\rangle_{A_1}$ ($|n\rangle_{A_2}$) and its symmetric counterpart $|n\rangle_{A_4}$ ($|n\rangle_{A_3}$). Thus the coefficients are related to each other through $d_{4n} = \kappa(-1)^n d_{1n}$ and $d_{3n} = \kappa(-1)^n d_{2n}$. This symmetry separates the state space into two different invariant subspaces can be labeled by the eigenvalues of the operation Π with $\kappa = +, -$. In accordance with the Schrödinger equation we find that the number of equations is reduced by half

$$\varepsilon_{1m} d_{1m} + \sum_{n=0}^{\infty} \Omega_{mn}^\kappa d_{2n} = E d_{1m}, \quad (4)$$

$$\varepsilon_{2m} d_{2m} + \sum_{n=0}^{\infty} W_{mn}^\kappa d_{1n} = E d_{2m}, \quad (5)$$

where $\varepsilon_{im} = (m - \beta_i^2) \omega_c$ and the off-diagonal terms describe the transitions between states belonging to different displaced Fock spaces

$$\begin{aligned} \Omega_{mn}^\kappa &= -\frac{\omega_1}{2} [A_1 \langle m|n\rangle_{A_2}] - \kappa(-1)^n \frac{\omega_2}{2} [A_1 \langle m|n\rangle_{A_3}], \\ W_{mn}^\kappa &= -\frac{\omega_1}{2} [A_2 \langle m|n\rangle_{A_1}] - \kappa(-1)^n \frac{\omega_2}{2} [A_2 \langle m|n\rangle_{A_4}]. \end{aligned}$$

Clearly the symmetry of the coefficients d_{im} reduces the number of equations by half. This effectively folds the already expanded Hilbert space to a diagonal block, in

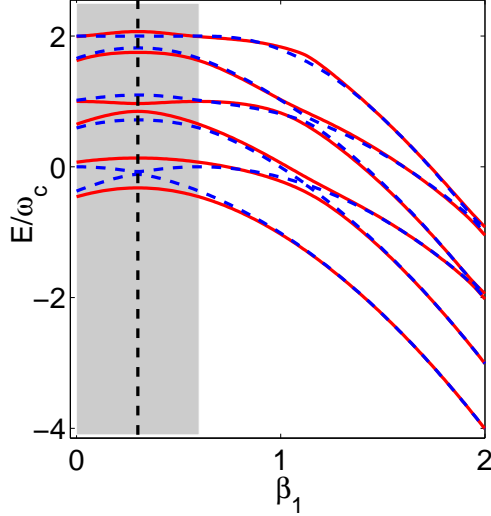


FIG. 1: (Color Online) The ED numerical solution of the energy levels as a function of coupling strength β_1 in the even (red solid lines) or odd (blue dashed lines) parity subspaces for $\omega_1 = \omega_2 = 0.25\omega_c$, $g_1 = 0.3\omega_c$. In the shadowed area the energy levels are symmetric about the vertical line $\beta_1 = 0.3$, i.e. $g_2 = 0$.

which the anti-diagonal elements are occupied by the newly added ω_N together with the parity κ as in the expressions for Ω and W above. The nonzero off-diagonal elements originates from the non-orthogonality of displaced Fock states [22]

$$A_i \langle m|n \rangle_{A_j} = e^{-\frac{\beta_{ij}^2}{2}} \sum_l^{\min(m,n)} \frac{(-1)^{n-l} \sqrt{m!n!}}{l!(n-l)!(m-l)!} \beta_{ij}^{m+n-2l} \quad (6)$$

with $\beta_{ij} = \beta_i - \beta_j$. This implies that the interchange of i and j , that of m and n , or the inversion of β_i to $-\beta_i$ will introduce a factor $(-1)^{m+n}$ in eq. (6). Then in terms of expressions about Ω_{mn}^κ and W_{mn}^κ , we find the symmetry relation

$$\Omega_{mn}^\kappa = W_{nm}^\kappa, \quad (7)$$

which turns the Hamiltonian in the Hilbert space into a real symmetric matrix which assures that all coefficients d_{im} are real. This allows us keep only Ω in the rest of the paper.

The 2^{N-1} equations for N qubits take similar forms as eqs. (4) and (5) with the diagonal terms $\varepsilon_{im}d_{im}$ and in each equation off-diagonal terms indicate the transitions between Fock states displaced in different directions and distances. It is easy to show that the eigenvalue spectrum of the N qubits system is unaltered when any of the coupling strengths changes its sign, e.g. $g_1 \rightarrow -g_1$ or $g_2 \rightarrow -g_2$ so that it suffices to discuss the energy spectrum for positive values of g_i . The equations (4) and (5) are solved by means of exact diagonalization (ED) and the energy spectrum are shown in Fig. 1 as functions of β_1 for $g_1 = 0.3\omega_c$ and $\omega_1 = \omega_2 = 0.25\omega_c$. As mentioned earlier, the power series (3) has to be truncated in order to obtain the spectrum. Here we set the truncation number $n_{tr} = 48$ such that the calculation is done in a closed subspace $|n\rangle_{A_i}$ with $(n = 0, 1, 2 \dots, n_{tr})$ and the off-diagonal elements Ω_{mn} for $m, n > n_{tr}$ are less than 10^{-6} . The lowest 6 levels are illustrated in Fig. 1 for even and odd parities respectively and we find similar to the single qubit case, strong coupling strength tends to lower the eigen-energies of the system and level crossing occurs for different parities, while levels with the same parity prefers to avoid this. Furthermore, the energy spectrum is symmetric about the vertical dashed line $\beta_1 = 0.3$ in the shadowed area (correspondingly g_2/ω_c takes the value between -0.3 and 0.3).

In any case, the process of getting analytically the eigenvalues and eigenvectors for the N -qubit system is very difficult, if not impossible, so approximation techniques have to be employed. Here we illustrate how to identify the building blocks of the determinant by taking the two-qubit system as an example, and the procedure of solving the eigen-equations up to any order of approximations to the exact result of the wave function. The condition for the existence of non-trivial solutions of d_{im} is the secular equation described by the determinant

$$\begin{vmatrix} \dots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & \varepsilon_{1m}-E & \Omega_{mm}^\kappa & 0 & \Omega_{m(m+1)}^\kappa & \dots \\ \dots & \Omega_{mm}^\kappa & \varepsilon_{2m}-E & \Omega_{(m+1)m}^\kappa & 0 & \dots \\ \dots & 0 & \Omega_{(m+1)m}^\kappa & \varepsilon_{1(m+1)}-E & \Omega_{(m+1)(m+1)}^\kappa & \dots \\ \dots & \Omega_{m(m+1)}^\kappa & 0 & \Omega_{(m+1)(m+1)}^\kappa & \varepsilon_{2(m+1)}-E & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \dots \end{vmatrix} = 0. \quad (8)$$

We see that in the two-qubit case the primitive building

block of the determinant is 2×2 and in the first-order

approximation two blocks m and $m + 1$ are involved, the transitions between which are determined by the off-diagonal elements $\Omega_{m(m+1)}$ and $\Omega_{(m+1)m}$. The primitive building block for N qubits is $2^{N-1} \times 2^{N-1}$, while higher-order approximation involves more identical blocks with m increased with step 1 and those off-diagonal Ω terms induce transition between blocks with different m .

A. Zeroth-order approximation

The key property of the inhomogeneously coupled N -qubit system is exhibited by the zeroth-order approximation of equation (8), which neglects all transitions between states with different m . This is often called the adiabatic approximation. Similar approximated solution to the JC model without the RWA is shown to be valid when the transition frequency of the qubit ω_0 is much smaller than the frequency of the single-mode bose field ω_c and it is very efficient for coupling strengths g up to or larger than the oscillator frequency [15]. For the N -qubit system we first consider the zeroth-order approximation and truncate the determinant Eq. (8) to the lowest order. This leaves us a block with the same index m , the diagonal terms of which read as $\varepsilon_{im} - E$ with $i = 1, 2, \dots, 2^{N-1}$ and the off-diagonal transition terms are those coefficients $A_i \langle m|m \rangle_{A_j}$. For two-qubit system the zeroth-order approximation for the determinant takes the following block form

$$\begin{vmatrix} \varepsilon_{1m} - E & \Omega_{mm}^\kappa \\ \Omega_{mm}^\kappa & \varepsilon_{2m} - E \end{vmatrix} = 0. \quad (9)$$

For convenience, we denote Ω_{mm}^κ as Ω_m^κ . Consequently the solutions for the eigenegies are

$$E_m^{\kappa\pm} = m\omega_c - (\beta_1^2 + \beta_2^2) \omega_c / 2 \pm \theta_m^\kappa \quad (10)$$

with $\theta_m^\kappa = \sqrt{(\Omega_m^\kappa)^2 + \omega_c^2 (\beta_1^2 - \beta_2^2)^2 / 4}$. Based on the symmetry of the coefficients d_{im} , the eigenstates of the system that satisfy the orthogonality and completeness conditions have the form

$$|\psi_m^{\kappa\pm}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} d_{1m}^{\kappa\pm} \\ d_{2m}^{\kappa\pm} \\ (-1)^m \kappa d_{2m}^{\kappa\pm} \\ (-1)^m \kappa d_{1m}^{\kappa\pm} \end{pmatrix}, \quad (11)$$

where

$$d_{1m}^{\kappa\pm} = \xi_m^{\kappa\pm} \sqrt{\frac{1}{1 + (\xi_m^{\kappa\pm})^2}}, d_{2m}^{\kappa\pm} = -\sqrt{\frac{1}{1 + (\xi_m^{\kappa\pm})^2}},$$

with $\xi_m^{\kappa\pm} = \Omega_m^\kappa / ((\beta_2^2 - \beta_1^2) \omega_c / 2 \mp \theta_m^\kappa)$.

A special case is the system with two completely identical qubits homogeneously coupled to a bose field, which means that $\beta_2 = 0$ and $\omega_1 = \omega_2$. This simplifies the transition frequency as $\Omega_m^\kappa \sim (1 + \kappa(-1)^m)$, i.e. the parity of

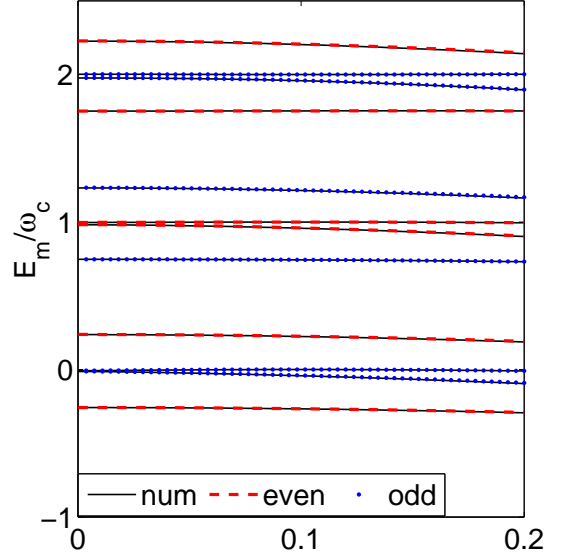


FIG. 2: (Color Online) The zeroth-order approximation of the energy levels with even and odd parities as a function of coupling strength g_2/ω_c for $\omega_1 = \omega_2 = 0.25\omega_c$ and $g_1 = 0.1\omega_c$, compared to the ED numerical ones.

the Hamiltonian (κ) and that of the displaced Fock space (m) together decide whether Ω_m^κ is zero or not. When $\Omega_m^\kappa = 0$ we have the eigenenergies

$$E_m^{\kappa+} = m\omega_c, E_m^{\kappa-} = (m - \beta_1^2) \omega_c, \quad (12)$$

and the corresponding eigenfunctions $|\psi_m^{\kappa+}\rangle = |\psi_1\rangle$, $|\psi_m^{\kappa-}\rangle = |\psi_2\rangle$, with

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, |\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}. \quad (13)$$

For nonzero Ω_m^κ we assume $|\Omega_m^\kappa|/\omega_c \gg \beta_1^2$ as in Ref. [41], the eigenegies can be expressed as

$$E_m^{\kappa+} = m\omega_c + \Omega_m^\kappa, E_m^{\kappa-} = m\omega_c - \Omega_m^\kappa, \quad (14)$$

with eigenfunctions $|\psi_m^{\kappa+}\rangle = |\psi_3\rangle$, $|\psi_m^{\kappa-}\rangle = |\psi_4\rangle$, with

$$|\psi_3\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, |\psi_4\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}. \quad (15)$$

We note that in writing down these four eigenstates of H the qubit basis are chosen as the uncoupled representation of spin operators $\hat{\sigma}_{1,2}^z$ in order to solve the qubits system with different frequencies and coupling strength. Three of states are alternatively [41] expanded in terms of the triplet states of the total spin component S_x of the two identical qubits in the case of homogeneous coupling. The spin singlet state is exactly ψ_1 in (13) which is itself the eigenstate of the Hamiltonian. By including the

singlet state into the eigenvectors, we are able to treat the dynamics of any initial state of the system. States such as $|10\rangle$ or $|01\rangle$, i.e. when the two qubits are respectively put in the upper and lower eigenstates of their $\hat{\sigma}^z$ s, are out of reach in the triplet manifold, which will be seen this in the next section. In Fig. 2, we can see that the analytical results in the zeroth-order approximation already agree with the exact solutions very well in the ultra-strong coupling case $g_2 \sim 0.2\omega_c$ ($g_1 = 0.1\omega_c$), where each four eigenenergies with the same index m bundle into a group corresponding to the four qubit basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ in the absence of coupling. In each group, the parity of each eigenstate is fixed with the lowest level being always even parity. For even m , two odd parity levels are held between two even parity ones, or vice versa for odd m . This can be compared to the single-qubit case where the energy levels are arranged as (even, odd), (odd, even), (even, odd), etc. because the qubit basis are instead $|0\rangle, |1\rangle$ and adding one photon will alter the parity.

B. First-Order Approximation

Next we consider the first-order approximation for the determinant (8) and permit transitions between blocks m and $m+1$ [22]. In the case of two-qubit system we consequently make a cut-off in Eq. (8) such that

$$\begin{vmatrix} \varepsilon_{1m} - E & \Omega_m^\kappa & 0 & \Omega_{m(m+1)}^\kappa \\ \Omega_m^\kappa & \varepsilon_{2m} - E & \Omega_{(m+1)m}^\kappa & 0 \\ 0 & \Omega_{(m+1)m}^\kappa & \varepsilon_{1(m+1)} - E & \Omega_{m+1}^\kappa \\ \Omega_{m(m+1)}^\kappa & 0 & \Omega_{m+1}^\kappa & \varepsilon_{2(m+1)} - E \end{vmatrix} = 0. \quad (16)$$

The equation can be solved analytically because it leads to a quartic equation in the form of $E^4 + bE^3 + cE^2 + dE + e = 0$. The coefficients in the quartic equation are assigned for parity κ and index m and expressed as (for simplicity we drop the superscript and subscript)

$$\begin{aligned} b &= -\varepsilon_{1m} - \varepsilon_{2m} - \varepsilon_{1(m+1)} - \varepsilon_{2(m+1)}, \\ c &= (\varepsilon_{1m} + \varepsilon_{2m})(\varepsilon_{1(m+1)} + \varepsilon_{2(m+1)}) \\ &\quad + \varepsilon_{1m}\varepsilon_{2m} + \varepsilon_{1(m+1)}\varepsilon_{2(m+1)} - (\Omega_m^\kappa)^2 \\ &\quad - (\Omega_{m+1}^\kappa)^2 - (\Omega_{(m+1)m}^\kappa)^2 - (\Omega_{m(m+1)}^\kappa)^2, \\ d &= (\varepsilon_{1m} + \varepsilon_{2(m+1)})(\Omega_{(m+1)m}^\kappa)^2 \\ &\quad + (\varepsilon_{1(m+1)} + \varepsilon_{2m})(\Omega_{m(m+1)}^\kappa)^2 \\ &\quad + ((\Omega_m^\kappa)^2 - \varepsilon_{1m}\varepsilon_{2m})(\varepsilon_{1(m+1)} + \varepsilon_{2(m+1)}) \\ &\quad + (\varepsilon_{1m} + \varepsilon_{2m})((\Omega_{m+1}^\kappa)^2 - \varepsilon_{1(m+1)}\varepsilon_{2(m+1)}), \end{aligned}$$

and

$$e = \begin{vmatrix} \varepsilon_{1m} & \Omega_m^\kappa & 0 & \Omega_{m(m+1)}^\kappa \\ \Omega_m^\kappa & \varepsilon_{2m} & \Omega_{(m+1)m}^\kappa & 0 \\ 0 & \Omega_{(m+1)m}^\kappa & \varepsilon_{1(m+1)} & \Omega_{m+1}^\kappa \\ \Omega_{m(m+1)}^\kappa & 0 & \Omega_{m+1}^\kappa & \varepsilon_{2(m+1)} \end{vmatrix}.$$

For each given parity κ and block index m we get in general four analytical solutions for the eigenenergies of Hamiltonian H

$$E_m^\kappa = -\frac{b}{4} \pm_\gamma Y \pm_s \frac{1}{2} \sqrt{-\left(4Y^2 + 2p \pm_\gamma \frac{q}{Y}\right)}, \quad (17)$$

where the two occurrences of \pm_γ must denote the same sign, while \pm_s can take its sign independently. The notations relating to the coefficients of the quartic equation are defined as $p = c - 3b^2/8$, $q = (b^3 - 4bc + 8d)/8$, $\Delta_0 = c^2 - 3bd + 12e$, $\Delta_1 = 2c^3 - 9bcd + 27b^2e + 27d^2 - 72ce$, and

$$Y = \frac{1}{2} \sqrt{-\frac{2p}{3} + \frac{Q + \Delta_0/Q}{3}},$$

$$Q = \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}.$$

The first-order approximation improves the analytical results applicable even in the deep coupling region $g > 1$ as shown in Fig. 3, where we set $g_1 = 0.3\omega_c$ and $\omega_1 = \omega_2 = 0.25\omega_c$. According to the assumption of the wavefunction (3), the dimension of the Hilbert space depends on the truncation of the displaced Fock number state as $4(n_{tr} + 1)$. Correspondingly, for each m we only have four genuine solutions. The zeroth-order approximation permits exactly four eigensolutions for each m as shown above, while in the first-order approximation one has eight eigensolutions for each combination $(m, m+1)$ including four even parity and four odd parity solutions.

Obviously the analytical results of first-order approximation show energy level crossing between states with the same parity or different parities. The level crossing of states with different parities are accidental and we will focus on those with the same parity. On the uncertainty whether or not energy levels can cross by changing parameters we give an argument in terms of von Neumann and Wigner non-crossing theorem: For a real symmetric (hermitian) matrix, we need to tune two (three) parameters to get a level crossing [57]. In the case of two-qubit system, the Hamiltonian in the Hilbert space is a real symmetric matrix. To determine the crossing or anti-crossing we calculate the fidelity between appropriate states before and after the crossing which is a measure of the "closeness" of two quantum states and defined as $F = |\langle\phi|\varphi\rangle|^2$ for pure states [58–62]. We remark that (1) for fixed m the energy levels can never cross by changing one parameter g_2 ; (2) However, for different m the energy level crossings may occur at some isolated points because two parameters m and g_2 are tuned. In the left (A) inset of Fig. 3, we zoom out a particular anti-crossing point of exact results of energy levels. Four analytical levels (deep-colored lines) in the first-order approximation match the exact results which avoid crossing, while the rest (light-colored lines) four curves mismatch and cross each other which should be discarded.

Taking the above aspects into consideration, for each invariant parity subspace we must rule out half solutions

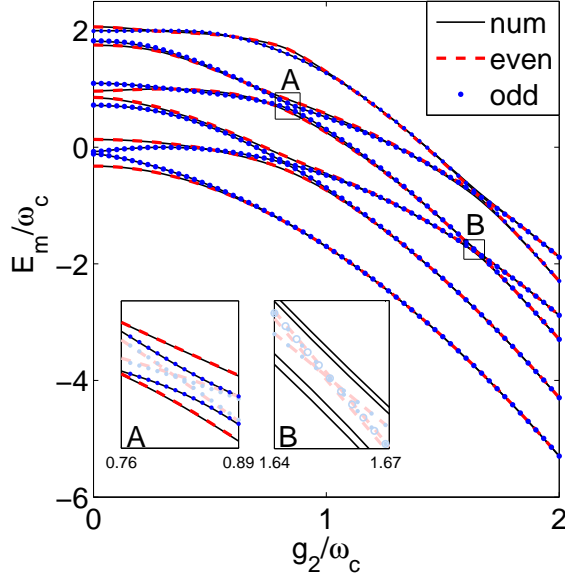


FIG. 3: (Color Online) The first-order approximation solution of the energy levels as a function of coupling strength g_2/ω_c with $\omega_1 = \omega_2 = 0.25\omega_c$ and $g_1 = 0.3\omega_c$, compared to the ED numerical ones. Insets show two typical level crossing points where the analytical results match the numerical ones (left) for $\beta_1 \sim 1$, or, on the other hand, where they mismatch (right) for even stronger coupling $\beta_1 \sim 2$. The pseudo-solutions in light-colored lines should be ruled out so that the analytical first-order results agree with the numerical ones perfectly.

of the first-order approximation by keeping only solutions (17) with \pm_γ opposite to \pm_s for each combination $(m, m+1)$ because each m has been used twice in our calculation. An exception is the combination $(0, 1)$, in which case we only drop the solution with both signs positive. In this way the pseudo solutions are removed and the analytical eigenvalues for the two-qubit system agree perfectly with the exact results in the deep coupling limit $g_2 \sim 1.5\omega_c$ ($g_1 = 0.3\omega_c$) in Fig. 3. The genuine solutions are two-fold degenerate in the deep strong coupling regime $g_2 > 1$. This degeneracy has been found in the quantum Rabi model for two qubits [39], though for both coupling parameters g_1, g_2 larger than the transition frequencies of the qubits.

Moreover, in the right inset (B) of in Fig. 3, as g_2 increases across the next level anti-crossing point all the solutions of the first-order approximation mismatch the exact results which suggests that higher order approximation is needed. Our procedure applies readily to the second-order approximation, in which case one has 12 solutions for each combination of three blocks $(m, m+1, m+2)$ and the degeneracy grows rapidly. In this way the approximated solutions will match the exact results in the right inset of Fig. 3. In particular, we notice that the transitions would be suppressed if the off-diagonal matrix elements are much smaller than the energy difference between the states belonging to differ-

ent blocks. Specifically, all elements $|\Omega_{nm}^\kappa|$, $|\Omega_{mn}^\kappa|$ with $n \neq m$ are set to zero in the zeroth-order approximation, while those with $n \neq m, m \pm 1$ are negligible in the first-order approximation.

C. Solvability and Integrability

In quantum mechanics there exist potentials for which it is possible to find a finite number of exact eigenvalues and associated eigenfunctions in the closed form. These systems are said to be quasi exactly solvable. The Rabi model is a typical example distinguished by the fact that part of its eigenvalues and corresponding eigenfunctions can be determined algebraically for special values of the energy splitting of the qubit ω and the coupling strength g [30–32]. Known as Judd’s isolated solutions [55], these exceptional spectrum with energy eigenvalues $E = n - g^2/\omega_c^2$ constitute the exact part of Rabi model and doubly degenerate with respect to parity.

Here we show that the two-qubit TC model provides another example of quasi-exactly solvable models, i.e part exact spectrum of the model can be obtained in some special parameter region. First of all, in the homogeneous coupling case $g_1 = g_2$, there always exists a constant solution $E = \omega_c$ corresponding to either the even parity eigenstates

$$|\psi_e\rangle = (q_e(|01\rangle - |10\rangle)|1\rangle + |11\rangle|0\rangle) / \sqrt{2q_e^2 + 1},$$

for the symmetric detunings with $\omega_1 + \omega_2 = 2\omega_c$ (suppose $\omega_1 > \omega_2$), or the odd parity eigenstates

$$|\psi_o\rangle = (q_o(|00\rangle - |11\rangle)|1\rangle + |01\rangle|0\rangle) / \sqrt{2q_o^2 + 1},$$

for the asymmetric detunings with $\omega_1 - \omega_2 = 2\omega_c$ with $q_{e,o} = 2g/(\omega_1 \mp \omega_2)$. Secondly, for two completely identical qubits homogeneously coupled to the bose field, i.e. $g_1 = g_2$ and $\omega_1 = \omega_2$, it is easy to prove the state $|\psi_1\rangle = (|10\rangle - |01\rangle)|m\rangle/\sqrt{2}$ in (13) for any m is exactly the eigenstate of H with eigenvalue $E_m = m\omega_c$. The state has even(odd) parity for odd(even) m . Very recently, an alternative form of analytical solution is given to the quantum Rabi models with two identical qubits in a similar way, however, essentially different from the Juddian solutions with doubly degenerate eigenvalues in the one-qubit quantum Rabi model [63]. In short, for the TC model with two qubits a finite part of the spectrum can be obtained in closed form and the remaining part of the spectrum is only numerically accessible. So we conclude that the TC model with two qubits is quasi exactly solvable.

In Fig. 4, we show the energy spectrum of homogeneous coupling system as a function of total coupling strength β_1 for $g_1 = g_2$ and $\omega_1 = \omega_2 = 0.25\omega_c$. In the decoupling limit $\beta_1 = 0$, the qubits are set free from the field, and we find the two odd(even) parity states, i.e. $|01\rangle|m\rangle$ and $|10\rangle|m\rangle$, are degenerate for even(odd) m , which now means the photon number of the free bosonic

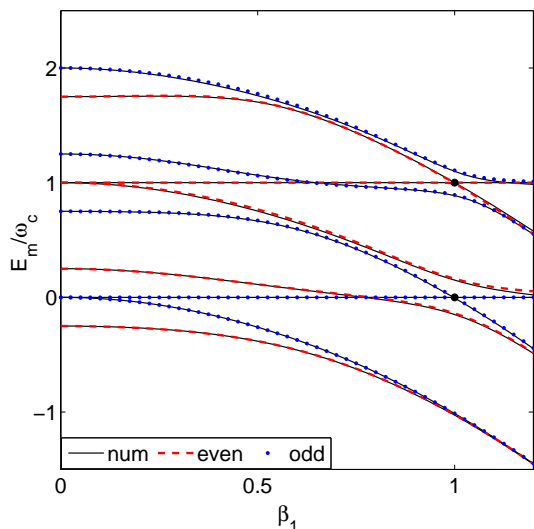


FIG. 4: (Color Online) The first-order approximation and numerical solution for the homogeneous coupling system ($g_1 = g_2$) as a function of total coupling strength β_1 with $\omega_1 = \omega_2 = 0.25\omega_c$. The two black points indicate the level crossing in the same parity subspace.

field. The first-order approximation is valid for the entire region $0 < \beta_1 < 1.2$ and it shows that the energy levels of the same parity would never cross with each other until $\beta_1 \sim 1$. The constant solutions $E_m = m\omega_c$ are shown as two horizontal lines $m = 0$ for odd parity and $m = 1$ for even parity in Fig. 4. Level crossing occurs in the same parity space when other solutions with different m sweep across them, denoted by two black dots at $\beta_1 \simeq 0.9898$ and 1.0004 . The fidelity of both the lower or upper states on the two sides of the crossing points is calculated to be exactly zero, which proves the existence of the level crossing. In Fig. 4 two parameters m and β_1 are tuned to get these level crossing points in the first-order approximation, which is consistent with the von Neumann and Wigner non-crossing theorem.

Besides those shown in Fig. 4, nontrivial level crossing points may appear in the fixed parity subspace for non-identical qubits. In the homogeneous coupling case ($g_1 = g_2$) a level crossing has been found in the even parity subspace for two inequivalent qubits with symmetric detunings $\omega_1 + \omega_2 = 2\omega_c$ [39] denoted as a black point in the upper panel of Fig. 4. We here show that this is only half of the story. It is actually found that the level crossing in the case of homogeneous coupling occurs for $\omega_1 \pm \omega_2 = 2\omega_c$. The reason is that the constant solution $E = \omega_c$ holds for either symmetric or asymmetric detuning conditions. Other levels would inevitably run across it with increasing coupling strength. In Fig. 5, we numerically confirm that the level crossing appears at $\beta_1 \simeq 1.0251$ for $\omega_1 = 1.3\omega_c, \omega_2 = 0.7\omega_c$ in even parity subspace [39], and at $\beta_2 \simeq 0.9442$ for $\omega_1 = 2.7\omega_c, \omega_2 = 0.7\omega_c$ in odd parity subspace. Level

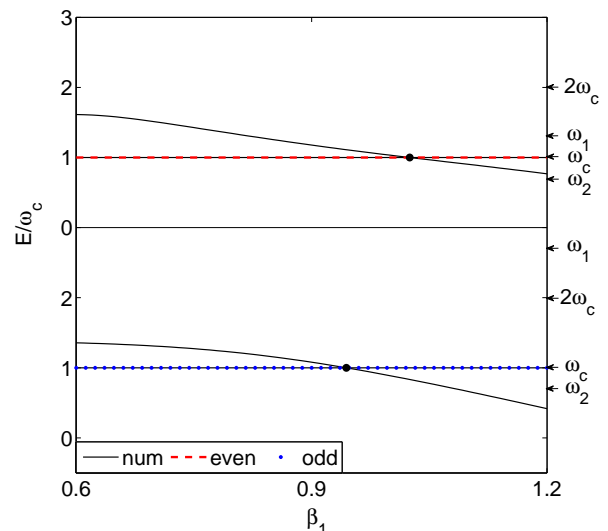


FIG. 5: (Color Online) Typical level crossing points for homogeneous coupled non-identical qubits with symmetric detuning $\omega_1 + \omega_2 = 2\omega_c$ in the even parity space (upper panel), and asymmetric detuning $\omega_1 - \omega_2 = 2\omega_c$ in the odd parity space (lower panel). The exact solutions $E = \omega_c$ are shown as red-dashed and blue-dotted horizontal lines. The frequencies of the qubits and the oscillator are labeled on the right side of the figure.

crossings in quantum theory are often related to symmetry. In the case of inhomogeneous coupling of nonidentical qubits with the oscillator mode no level crossing in the same parity space is found, while $g_1 = g_2$ enlarges the symmetry so that the level crossing appears on the line $E = \omega_c$, and $\omega_1 = \omega_2$ enlarges the symmetry even further and more crossing points appear on $E = m\omega_c$. This is similar to what happened in the JC model - the enlarged symmetry by RWA leads to two level crossing points in the even parity space (Fig. 3 in [26]) which are not found in the Rabi model.

The appearance of level crossing and associated symmetry are essentially related to the integrability of our model, which we shall address in the following. According to the criterion on quantum integrable proposed by Braak [26], integrability is equivalent to the existence of f numbers to classify the eigenstates uniquely with f the sum of discrete and continuous d.o.f. The Rabi model has two d.o.f. and at the same time can be uniquely labelled by two quantum numbers associated with the energy level and the parity, respectively. So the Rabi model is considered to be integrable [26, 33]. We argue that the homogeneous coupled two-qubit TC model with $\omega_1 = \omega_2$ or $\omega_1 \pm \omega_2 = 2\omega_c$ is integrable, based on the following three reasons:

(1) In general a system is not integrable in the whole parameter region, but it may be integrable under some special conditions. A generalization of Rabi model with an additional term $\epsilon\sigma_x$ (Eq. (7) of Ref. [26]) breaking the

parity symmetry is constructed to be the first example of nonintegrable but exactly solvable system. Clearly we recover the integrable Rabi model for $\epsilon = 0$. Another example is spinor Bose-Einstein condensate of alkali gases [64]. Generally considered to be nonintegrable, the 1D homogeneous spinor bosons can be exactly solved with Bethe ansatz (BA) method [65] along two integrable lines $c_2 = 0$ and $c_0 = c_2$. In spite of the nonintegrability of our model in the whole parameter regime, there do exist special situation for the parameters, i.e. $\omega_1 = \omega_2$ or $\omega_1 \pm \omega_2 = 2\omega_c$, in which case the homogeneous coupled TC model becomes integrable.

(2) The system is nonintegrable if the total number of d.o.f. exceeds the quantum numbers used to label the eigenstates uniquely. In the case of Braak's generalized Rabi model the absence of any level crossing in the spectral graph is sufficient to rule out its integrability. There are two d.o.f and one quantum number, energy, is sufficient to number the eigenstates uniquely. In the case of two-qubit TC model, for the simplified case of $\omega_2 = 0$ no level crossing is found in the spectra graph [33], while for inhomogeneous coupling accidental level crossing occurs only for different parities (Figure 1). In both cases the model is nonintegrable because we have three d.o.f and the two quantum numbers, energy and parity, are enough to label the states uniquely.

(3) Level crossing in the same parity subspace is non-trivial because we need another quantum number other than parity to label the degenerate states, which leads the system to be integrable. As a typical integrable system, each eigenstate in the hydrogen atom is assigned three quantum numbers n, l, m which characterize the quantization of radial, angular and orientation of the electronic orbit. None of them can be omitted in picking up the state of this three d.o.f system. This parallels the characterization of each eigenstate of Rabi model through two quantum numbers, the parity quantum number n_0 and the n_1 th zero of the transcendental function $G(x)$, corresponding to two d.o.f. For JC model with enlarged $U(1)$ symmetry, level crossing occurs in the same parity space. However, we are lucky that the operator C can be used for a further decomposition of the subspaces with fixed parity. The state spaces entails a second possibility to label the states uniquely through C and a two-valued index n_0 , with the parity being a redundant quantum number. Similarly, the level crossing appeared in homogeneous coupled two-qubit TC model implies an enlarged hidden symmetry for $\omega_1 = \omega_2$ or $\omega_1 \pm \omega_2 = 2\omega_c$. What we need to do is to find a C -like conserved quantity to decompose the even and odd subspaces further. Consequently we would have three quantum numbers (parity, two-valued index n_0 and C -like number) to uniquely label the state with three d.o.f. Though not an exact result, in the zeroth and first order approximation we have shown a possible scheme of labeling the states with parity, \pm and m . In summary the homogeneous coupled model is integrable for two identical qubits or with (a)symmetric detuning, though further exploration of the

conserved quantity and hidden symmetry is needed.

III. POPULATION INVERSION DYNAMICS

To better learn the quantum behavior in the prototypical problem of cavity electrodynamics with more than one qubit involved, we study the dynamical properties of a two-qubit system strongly coupled to a high-frequency quantum oscillator. The eigenvectors and eigenvalues of the system derived in Sec. II can be taken as a complete set, upon which the time evolution of wave function can be expanded. We discuss here the probability of finding the two qubits remaining in the initial state [15], which is essentially the fidelity between the wave function at subsequent time t and the initial state.

The simplest dynamical behavior is considered when we put the qubits initially in any one of the four product states and the initial state of the oscillator is prepared in the displaced Fock basis corresponding to them. In the zeroth-order approximation, these initial states are inversely linear combinations of the eigenvectors of the Hamiltonian (11) and expressed respectively as following

$$\begin{aligned} |11\rangle |m\rangle_{A_1} &= \frac{1}{\sqrt{2}} \sum_{\kappa, \gamma=\pm} d_{1m}^{\kappa\gamma} |\psi_m^{\kappa\gamma}\rangle \\ |10\rangle |m\rangle_{A_2} &= \frac{1}{\sqrt{2}} \sum_{\kappa, \gamma=\pm} d_{2m}^{\kappa\gamma} |\psi_m^{\kappa\gamma}\rangle \\ |01\rangle |m\rangle_{A_3} &= \frac{1}{\sqrt{2}} \sum_{\kappa, \gamma=\pm} (-1)^m \kappa d_{2m}^{\kappa\gamma} |\psi_m^{\kappa\gamma}\rangle \\ |00\rangle |m\rangle_{A_4} &= \frac{1}{\sqrt{2}} \sum_{\kappa, \gamma=\pm} (-1)^m \kappa d_{1m}^{\kappa\gamma} |\psi_m^{\kappa\gamma}\rangle, \end{aligned}$$

which in the special case of two completely identical qubits reduce to

$$\begin{aligned} |11\rangle |m\rangle_{A_1} &= \frac{1}{2} (\sqrt{2} |\psi_2\rangle + |\psi_3\rangle + |\psi_4\rangle) \\ |10\rangle |m\rangle_{A_2} &= \frac{1}{2} (\sqrt{2} |\psi_1\rangle - |\psi_3\rangle - |\psi_4\rangle) \\ |01\rangle |m\rangle_{A_3} &= -\frac{1}{2} (\sqrt{2} |\psi_1\rangle - |\psi_3\rangle - |\psi_4\rangle) \\ |00\rangle |m\rangle_{A_4} &= -\frac{1}{2} (\sqrt{2} |\psi_2\rangle + |\psi_3\rangle + |\psi_4\rangle). \end{aligned}$$

As an example, we study the system dynamics with only one qubit, say qubit 2, being excited to the upper level, i.e. $\Psi(0) = |10\rangle |m\rangle_{A_2}$. The probability of finding the two qubits in any possible product states is easily obtained and we are interested in the fidelity to the initial state

$$P_{10}(m, t) = |_{A_2} \langle m | \langle 10 | \Psi(t) \rangle|^2 \quad (18)$$

with

$$\Psi(t) = \frac{1}{\sqrt{2}} \sum_{\kappa, \gamma=\pm} d_{2m}^{\kappa\gamma} |\psi_m^{\kappa\gamma}\rangle e^{-iE_m^{\kappa\gamma} t}.$$

It is easy to show that $\xi_m^+ \xi_m^- = -1$, $(d_{1m}^{\kappa+})^2 = (d_{2m}^{\kappa-})^2$. By means of these, we find the probability of the two qubits staying in their initial states consists of four oscillating terms, the frequencies of which are all possible combinations of θ_m^\pm , i.e.

$$P_{10}(m, t) = \frac{1}{2} \left\{ 1 + \sum_{\kappa=\pm} (c_{1m}^\kappa)^2 (\cos(2\theta_m^\kappa t) - 1) + (1 - c_{2m}) \cos(\theta_m^+ - \theta_m^-) t + c_{2m} \cos(\theta_m^+ + \theta_m^-) t \right\} \quad (19)$$

with the two coefficients defined as

$$c_{1m}^\kappa = \frac{\xi_m^{\kappa+}}{1 + (\xi_m^{\kappa+})^2},$$

$$c_{2m} = \frac{(\xi_m^{++})^2 + (\xi_m^{-+})^2}{(1 + (\xi_m^{++})^2)(1 + (\xi_m^{-+})^2)}.$$

For homogeneous coupling $\beta_2 = 0$ and $\omega_1 = \omega_2$, eq. (19) is reduced to

$$P_{10}(m, t) = \frac{1}{8} \{ 2 + \cos(2\Omega_m^+ t) + \cos(2\Omega_m^- t) + 2(\cos(\Omega_m^+ + \Omega_m^-) t + \cos(\Omega_m^+ - \Omega_m^-) t) \},$$

which consists essentially two oscillating terms because $\Omega_m^+ = 0$ for even m and $\Omega_m^- = 0$ for odd m .

Consider now the harmonic oscillator in the state which most closely approaches the classical limit, that is, we choose the oscillator begins in the displaced coherent state. The initial state is thus given by

$$|\Psi(0)\rangle = \sum_{m=0}^{\infty} \frac{e^{-|z|^2/2} z^m}{\sqrt{m!}} |10\rangle |m\rangle_{A_2}. \quad (20)$$

The probability of two qubits remaining in their initial state $|10\rangle$ is calculated by tracing over all Fock states of the oscillator as follows

$$P_{10}(z, t) = \langle 10 | \text{Tr} \rho(z, t) | 10 \rangle = \sum_{m=0}^{\infty} p(m) P_{10}(m, t), \quad (21)$$

where $\rho(z, t) = |\Psi(t)\rangle \langle \Psi(t)|$ is the density matrix of the system and the normalized Poisson distribution is defined as

$$p(m) = \frac{e^{-|z|^2} |z|^{2m}}{m!}.$$

We find that the probabilities of two qubits populating in the four product states oscillate with the same characteristic frequencies. For the states $|10\rangle$ or $|01\rangle$ the oscillation is around an equilibrium position $(1 - B)/2$, while for the states $|11\rangle$ or $|00\rangle$ the oscillation equilibrium is $B/2$ with $B = \sum_{m=0}^{\infty} \sum_{\kappa=\pm} p(m) (c_{1m}^\kappa)^2$. For the homogeneous coupling case and $\omega_1 = \omega_2$, we recover the analytical

result established previously [41] by keeping only three terms $l = m, m - 1, m - 2$ in the summation of Ω_m^\pm (eq. 6) and replacing the Poisson distribution by a Gaussian one for big enough $|z|$

$$P_{10}(z, t) = \frac{3}{8} + \frac{1}{2} S(t, \omega_1) + \frac{1}{8} S(t, 2\omega_1), \quad (22)$$

where

$$S(t, \omega_1) = \text{Re} \left[\sum_{k=0}^{\infty} \bar{S}_k(t, \omega_1) \right]$$

and

$$\bar{S}_k(t, \omega_1) = \frac{\exp(\Phi_{Re} + i\Phi_{Im})}{(1 + (\pi k f)^2)^{1/4}} \quad (23)$$

with

$$\Phi_{Re} = \frac{-(\mu - \mu_k)^2 f \beta_1^2}{2(1 + (\pi k f)^2)},$$

$$\Phi_{Im} = \frac{\tan^{-1}(\pi k f)}{2} + \mu - |z|^2 (\mu \beta_1^2 - 2\pi k).$$

Here we have defined $f = |z|^2 \beta_1^2$, $\mu = \omega_1 t e^{-\beta_1^2/2}$, $\mu_k = 2\pi k(1 + f/2)/\beta_1^2$. It is obviously that the revival in $S(t, \omega_1)$ occurs around each time $\mu = \mu_k$, the envelope and the fast oscillatory of which are determined by the exponential and cosine terms respectively.

We restrict our discussion to the regime of large detuning $\omega_c \gg \omega_j$ and ultrastrong coupling strength $g_j \sim 0.1\omega_c$. Fig. 6 shows the results of the time evolution of the probability (21) by means of the zeroth-order approximation analytical method compared with the numerically exact solution, where we have made a cutoff for m to a maximum value 30 because $p(m) \approx 0$ for $z = 3$ and $m \geq 30$. Our approximated results prove to be unexpectedly powerful, giving accurate dynamics perfectly in the present experimentally accessible coupling regime. Here we assume that one of two qubits reaches ultrastrong coupling regime $g_1/\omega_c = 0.1$, while the coupling strength to the other qubit (g_2/ω_c) changes from small to large. In Fig. 6(a) g_2 is much smaller than g_1 and there is no collapse-revival phenomena in the evolution of the probability. As g_2 increases, the collapses and revivals emerge gradually and the first collapse becomes faster and faster and the revival signal is more and more distinct in Fig. 6(b-c). In Fig. 6(d) the two coupling strengths are equal, resulting in the most regular shape in the oscillation of probability and the peaks become periodic in time. Finally in Fig. 6(e) when g_2 is larger than g_1 , collapses and revivals continue for a while and the oscillation becomes apparently irregular. Because $(c_{1m}^\kappa)^2 < c_{2m}$, the revivals with smaller amplitude are mainly determined by the second term of Eq. (19) containing c_{1m}^κ , instead, the revivals with larger amplitude depend on the third and fourth terms containing c_{2m} . The above analysis and

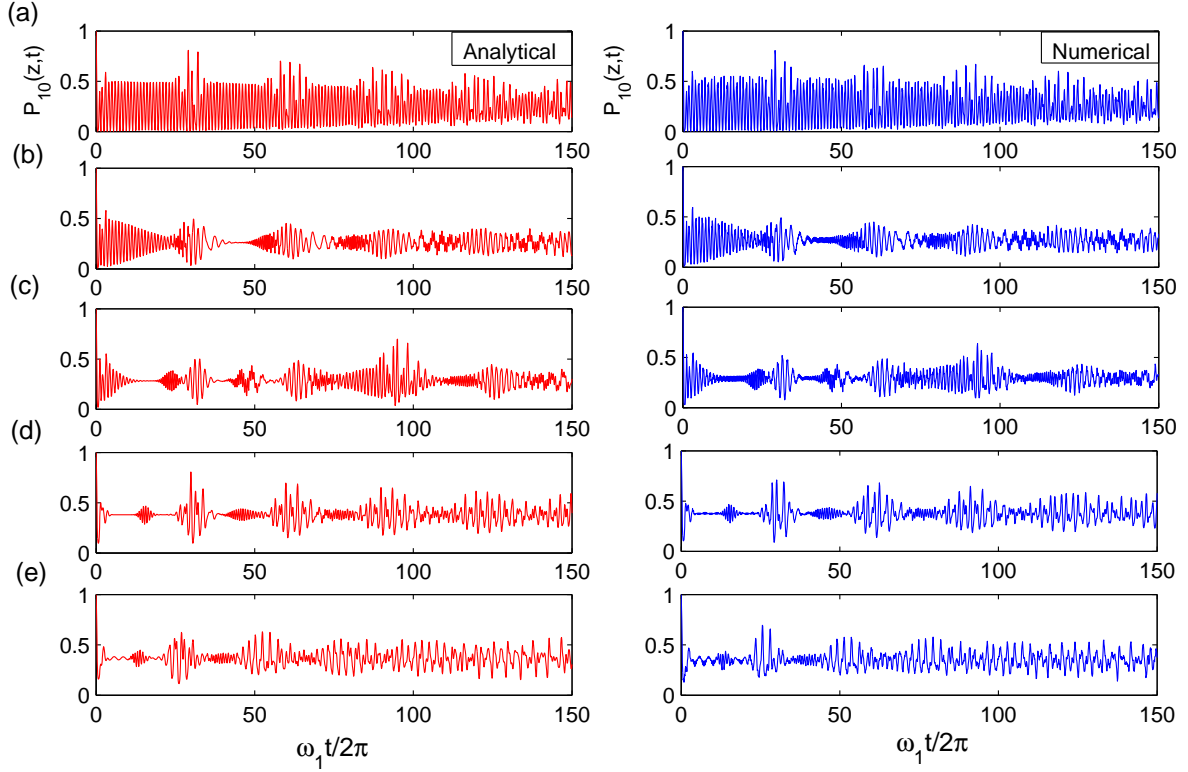


FIG. 6: (Color Online) Probability $P_{10}(z,t)$ of finding two qubits in state $|10\rangle$ as a function of $\omega_1 t/2\pi$, by the zeroth-order approximation analytical method and the numerical result for different coupling strengths. The zeroth-order approximation presents a pretty good description of the time-dependent evolution of the probability. We consider two identical qubits with $\omega_1 = \omega_2 = 0.15\omega_c$ and g_1 is fixed to $0.1\omega_c$. The coupling strength to the qubit 2 is $g_2 = 0.01\omega_c$ (a), $0.03\omega_c$ (b), $0.05\omega_c$ (c), $0.1\omega_c$ (d) and $0.12\omega_c$ (e), respectively. The oscillator is prepared initially in its coherent state with $z = 3$.

discussions suggest that the collapse-revival phenomena are sensitive to the coupling strength in the evolution of the probability, as well as it is periodic only for $g_1 = g_2$ and $\omega_1 = \omega_2$. We also find that the probability of two qubits populating in the four product states exhibit similar envelope of the revival signal.

It is sometimes more convenient to measure the population inversion in one of the qubit, that is, we observe the time evolution of the expectation value of the Pauli matrix operator of qubit 1 defined as $\sigma_1^z = (|1\rangle\langle 1| - |0\rangle\langle 0|)_1$ and related to the probabilities through $\langle \sigma_1^z \rangle = P_{11}(z,t) + P_{01}(z,t) - P_{10}(z,t) - P_{00}(z,t)$. In doing this, we again fix the value of $g_1 = 0.1\omega_c$ and study how the existence of qubit 2 will change the dynamics of the qubit 1. With the initial state (20) the expectation value of σ_1^z is calculated as

$$\langle \sigma_1^z \rangle = \sum_{m=0}^{\infty} p(m) \{ (D_m - 1) \cos(\theta_m^+ - \theta_m^-) t - D_m \cos(\theta_m^+ + \theta_m^-) t \}, \quad (24)$$

with $D_m = 2c_{1m}^+ c_{1m}^- + c_{2m}$. In the case of $\beta_2 = 0$ this reduces to

$$\langle \sigma_1^z \rangle = -S(t, \omega_1). \quad (25)$$

In Fig. 7 we show the time dependent inversion of qubit 1 for different coupling strength between qubit 2 and the bose field. Due to the excellent agreement with the numerically exact solution, we only show the analytical result in Fig. 7 and the parameters are the same as in Fig. 6. In the absence of qubit 2, the dynamics of a single qubit already exhibits the collapse-revival phenomena in the strong coupling regime $g_1 \sim 0.1$. The population inversion given in (24) shows that the revival signal is robust for weak coupling to the second qubit - we even can not tell the difference for the single qubit dynamics and that for a coupling strength $g_2 = 0.01$ and the revival signals for weak coupling cases are only found to be delayed a little in Fig. 7(a-c). With the increasing of g_2 to the same amplitude as g_1 the revival signal is destroyed, indicating that the qubit 2 influences the qubit 1 by interacting with the optical field. This behavior can be understood qualitatively as following. For each m , besides a common factor $p(m)$ Eq. (24) consists now of two cosine terms, whose amplitudes are determined by D_m . For $g_2 < g_1$, we can show numerically that D_m is always smaller than 0.5 and can be neglected for larger m . The dynamics depends thus mainly on the difference, instead of the summation, of θ_m^\pm as in the first term in (24). This

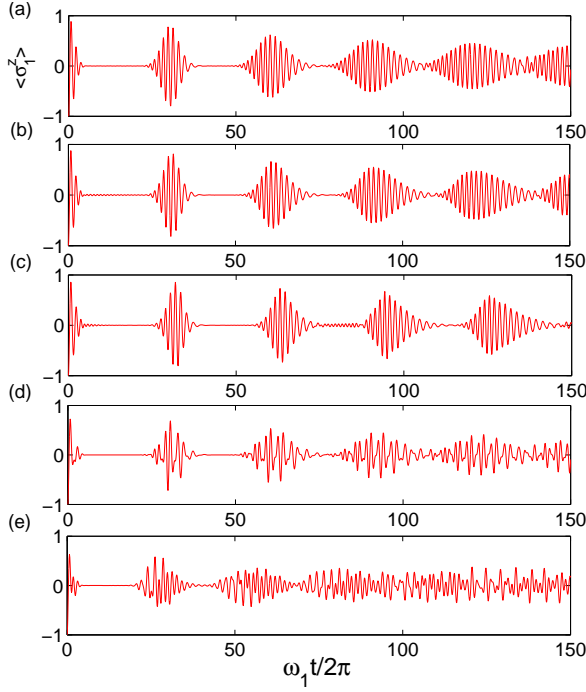


FIG. 7: (Color Online) The time-dependent inversion of qubit 1 as a function of $\omega_1 t / 2\pi$ for different coupling strength between qubit 2 and a bose field, given by our zeroth-order approximation method. The corresponding parameters are the same as in Figure 6.

gives the periodicity of revivals in Fig. 7, which would persist even for the homogeneous coupling case when the two terms in (24) are comparable. In Fig. 7(e) the interference of the two revival signal terms with almost equal amplitude leads to the irregular oscillation of population inversion.

IV. ENTANGLEMENT BEHAVIORS

Quantum entanglement can be used in studies of fundamental quantum phenomena and the on-chip entanglement of solid-state qubits provides a key building block for the solid-state realization of quantum optical networks. It has attracted much attention in connection with Bell's inequality [50, 51, 53]. However, realization of long-distance entanglement based on solid-state systems coupled to an optical field is an outstanding challenge. In the homogeneous coupling case with equal strengths to two identical qubits the entanglement properties have recently been studied [41]. In this section we aim to describe the entanglement properties by considering different coupling strengths to the two qubits. Thus it would be very interesting to study more general quantum correlations between two qubits. We suppose an initial entanglement of the two qubits in the form of a familiar Bell state and the oscillator in a coherent state, which is

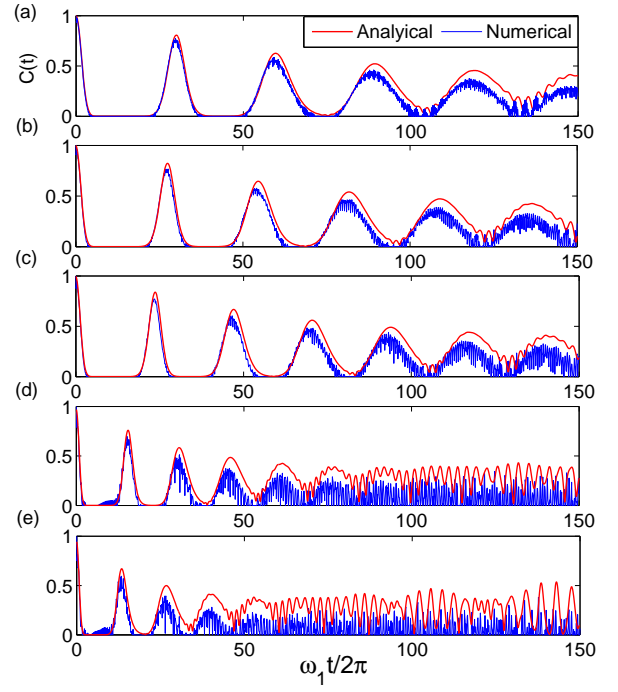


FIG. 8: (Color Online) Plots of the concurrence evolutions as a function of $\omega_1 t / 2\pi$ for different coupling strength between qubit 2 and a bose field, given by our zero-order approximation approach and numerical method. The corresponding parameters are the same as in Figure 6.

expressed as

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}} (|11\rangle + |00\rangle) |z\rangle. \quad (26)$$

As a good approximations in the case of small β_1 we may expand the state $|m\rangle$ in terms of the displaced Fock space and the most important contribution in the summation over m are the terms with the same m , which is equivalent to take $|m\rangle \approx |m\rangle_{A_i}$ [41]. Thus we can obtain

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}} \sum_{m=0}^{\infty} \frac{e^{-|z|^2/2} z^m}{\sqrt{m!}} (|11\rangle |m\rangle_{A_1} + |00\rangle |m\rangle_{A_4}). \quad (27)$$

The initially entangled state of two qubits evolves into $|\Psi(t)\rangle$ which is given by

$$|\Psi(t)\rangle = \sum_{m=0}^{\infty} \sum_{\kappa=\pm} \left(e^{-iE_m^{\kappa\gamma} t} |\psi_m^{\kappa\gamma}\rangle \langle \psi_m^{\kappa\gamma} | \Psi(0) \rangle \right). \quad (28)$$

To quantify the entanglement of a two-qubit system, we need to calculate the reduced density operator by trac-

ing out the quantum field. The result is given by

$$\begin{aligned}\hat{\rho}_Q(t) &= \sum_m \langle m | \Psi(t) \rangle \langle \Psi(t) | m \rangle \\ &= \sum_{m=0}^{\infty} \sum_{\kappa=\pm} \frac{p(m)(1+\kappa(-1)^m)}{4} (q_{1m}^{\kappa+} |ee\rangle \langle ee| \\ &\quad + q_{2m}^{\kappa+} |ee\rangle \langle gg| + q_{2m}^{\kappa-} |gg\rangle \langle ee| + q_{1m}^{\kappa-} |gg\rangle \langle gg|)\end{aligned}\quad (29)$$

where the calculation is done in the eigenbasis $|e\rangle$ and $|g\rangle$ of σ^x with eigenvalues $\pm 1/2$ respectively. The coefficients q 's are defined as

$$\begin{aligned}q_{1m}^{\kappa\pm} &= 1 \mp \frac{4\xi_m^{\kappa+} \left((\xi_m^{\kappa+})^2 - 1 \right) \sin^2(\theta_m^{\kappa} t)}{\left((\xi_m^{\kappa+})^2 + 1 \right)^2}, \\ q_{2m}^{\kappa\pm} &= 1 - \frac{8(\xi_m^{\kappa+})^2 \sin^2(\theta_m^{\kappa} t)}{\left((\xi_m^{\kappa+})^2 + 1 \right)^2} \pm i \frac{2\xi_m^{\kappa+} \sin(2\theta_m^{\kappa} t)}{(\xi_m^{\kappa+})^2 + 1},\end{aligned}$$

which are all unity at $t = 0$. This means that the qubits are initially prepared in a pure state, but as time evolves, the reduced state of the qubits becomes mixed. Obviously the reduced density matrix in the eigenspace of the spin product operators $\sigma_1^x \otimes \sigma_2^x$ with the standard two-qubit basis $|ee\rangle, |eg\rangle, |ge\rangle, |gg\rangle$ belongs to a special class of density matrices (X-matrices) with only diagonal and anti-diagonal elements. It is thus more convenient to quantify the entanglement using concurrence, which in our case takes a very simple form [41, 51]

$$C(t) = \left| \sum_{m=0}^{\infty} \sum_{\kappa=\pm} \frac{p(m)(1+\kappa(-1)^m)}{2} q_{2m}^{\kappa+} \right|. \quad (30)$$

It is also worthwhile to mention that for homogeneous coupling $\beta_2 = 0$ we immediately have $q_{1m}^{\kappa\pm} = 1$ and $q_{2m}^{\kappa\pm} = e^{\pm i 2\Omega_m^{\kappa} t}$. Then the concurrence (30) reduces to the homogeneous result as in [41]

$$C(t) \approx \sum_{k=0}^{\infty} |\bar{S}_k(t, 2\omega_1)| = \sum_{k=0}^{\infty} \frac{\exp\left(\frac{-(2\mu - \mu_k)^2 f \beta_1^2}{2(1 + (\pi k f)^2)}\right)}{(1 + (\pi k f)^2)^{1/4}}.$$

In Fig. 8 we plot the time evolution of the concurrence of two qubits, coupled to the bose field with different coupling strengthes. It is interesting to examine how the entanglement changes when one of two qubits reaches the ultrastrong coupling regime while the other coupling parameter varies. Obviously, we observe a variety of qualitative features such as entanglement birth, death, as well as rebirth, in which revivals appear periodically. Moreover, the periodicity of revivals disappears after a period of time and the duration of death time becomes shorter and shorter over time. We realize that the period gets shorter and shorter and the periodicity of revivals vanishes faster and faster with the increasing

of g_2 . The zero-order approximation reproduces quite accurate evolution in short time and fails in describing the long time behavior when the coupling strengthes are sufficiently large.

V. CONCLUSION

In conclusion, we have developed a systematic truncated subspace approach for solving the TC model beyond RWA by using the displaced Fock basis, parity operator subspace and truncation in the power series [15, 22, 26]. This provides a straightforward way to access the Hilbert space of the inhomogeneous coupling system. In principal we are able to solve the inhomogeneously coupled N -qubits-oscillator model to get an analytical result to any order by constructing the 2^N displacement operators. The complexity of the solutions depends on the determinant of the secular equation, the primitive building blocks of which involve transition between Fock states displaced in different directions and distances. As an example of particular experimental interest, the two-qubit TC model manifests a lot of new features of the qubit-oscillator system and our main findings include:

(1) The analytical energy spectrum of the two-qubit inhomogeneous coupling TC model are given in the zeroth-order and first-order approximations. The zeroth-order results already agree with the numerical solutions very well in the ultra-strong coupling case $\beta_1 \sim 0.2\omega_c$, while the first-order approximation improves the analytical eigenenergies applicable even in the deep coupling regime $\beta_1 \sim \omega_c$ after half of the pseudo solutions are ruled out.

(2) The TC model consisting of two qubits is quasi-exactly solvable, that is, a finite number of exact eigenvalues and associated eigenfunctions are given in the closed form. Specifically, in the homogeneous coupling case, $E = \omega_c$ is always a solution corresponding to even(odd) parity for symmetric(asymmetric) detuning $\omega_1 \pm \omega_2 = 2\omega_c$. For two completely identical qubits homogeneously coupled to the bose field, the singlet state $(|10\rangle - |01\rangle)|m\rangle/\sqrt{2}$ for any m is an exact eigenstate with eigenvalue $E_m = m\omega_c$. The remaining part of the spectrum is only numerically accessible through truncation subspace approach.

(3) Several nontrivial level crossing points in the same parity subspace are identified by means of the fidelity between states before and after the crossing. This implies an enlarged hidden symmetry and we show that the homogeneous coupled two-qubit TC model with $\omega_1 = \omega_2$ or $\omega_1 \pm \omega_2 = 2\omega_c$ is integrable.

(4) The quantum dynamical of the TC model beyond the RWA are investigated in the adiabatic approximation, with a special attention paid on the unequal coupling strengths for the two qubits. The probability of the two qubits staying in their initial states is characteristic of four oscillating frequencies, which is distinct from that of the single qubit system and the homogeneous coupling system. The approximated results of population inver-

sion are surprisingly accurate in describing the dynamics of the qubit, which shows that the collapse-revival phenomena emerge, survive, and are finally destroyed when the coupling strength increases beyond the deep coupling regime. This provides a method to control the revival signal of one qubit by means of the involvement of another one, which imprints its influences in the system by interacting with the optical field.

(5) The entanglement evolution of the two qubits as a principal measure of intrinsically quantum coherence is examined with an initial inter-qubit entanglement in the form of a familiar Bell state and the oscillator in a coherent state. Analytical results are obtained for the concurrence in the inhomogeneous coupling case by tracing out the quantum field in the reduced density matrix.

Our approximation approach is applicable to systems of arbitrary two qubits satisfying $(|g_1| + |g_2|) \leq 0.2\omega_c$ and $\omega_c \gg \omega_j$. The time evolution of the two qubits reproduces perfectly the special case with two completely iden-

tical qubits homogeneously coupled to a common oscillator mode, i.e. $g_1 = g_2$ and $\omega_1 = \omega_2$ as in Ref. [41]. Interestingly, there are still further work to do in the multiple qubits and oscillator system in the ultrastrong regime, e.g. the GHZ state entanglement evolution, quantum entanglement between the polarization of a single optical photon and solid-state qubits, the decoherence behavior analysis in an external environment, etc.

Acknowledgments

This work is supported by the NSF of China under Grant Nos. 11234008, 11104171 and 11074153, the National Basic Research Program of China (973 Program) under Grant Nos. 2010CB923103, 2011CB921601. We thank D. Braak, Tao Liu, Yuxi Liu, Li Wang and Qinghu Chen for helpful discussions.

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